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Constrained Hamiltonian systems as implicit differential equations

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Abstract. Generalized Hamiltonian systems derived from degenerate Lagrangians are presented as implicit differential equations rather than families of Hamiltonian vector fields restricted to constraint subsets of the phase space. Systems considered are more general than the ones described by Dirac—they are generated by Morse families of functions and not by constrained Hamiltonians. Integrability of such systems is analysed. Examples are given.

1. Introduction

First attempts to deal with Hamiltonian systems generated by degenerate Lagrangians were made by Dirac and Bergmann [2]. A complete solution of the problem of singular Lagrangians was not given—the Legendre transformation for singular Lagrangians was not constructed. A complete solution of the problem was found in 1974 [9]. Generalized Hamiltonian systems were introduced as implicit differential equations and the Legendre transformation for singular Lagrangians was defined. It turns out that the Legendre transformation applied to a singular Lagrangian rarely results in a generalized Hamiltonian system characterized by a Hamiltonian defined on a constraint submanifold considered by Dirac. More general Hamiltonian systems have to be considered. These systems introduced in [9] are called *generalized Dirac systems* in the present paper. Examples of integrable generalized Dirac systems were given in [7]. In the present paper an algorithm for extracting the integrable part of an implicit differential equation is formulated and adapted to generalized Dirac systems generated by Morse families. This work follows an earlier paper on integrability of Dirac systems [6]. There is a minor change in the formulation of the general integrability algorithm in section 6 and the integrability of Dirac systems is discussed in terms of Morse families. Few non-integrable mechanical systems are known. Only one of our examples in the last section is derived from physics. Implicit differential equations related to generalized Dirac systems were introduced by Marle [5] and Dazord [1].

2. Preliminary definitions

The reader is assumed to be familiar with basic concepts of differential geometry. We review below the definitions of concepts fundamental for the discussion of implicit differential equations and their integrability.

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The *tangent bundle* of a differential manifold M of dimension m is a differential manifold $\mathsf{T}M$ of dimension $2m$. The underlying set of $\mathsf{T}M$ is the set of equivalence classes of differentiable curves in M called *vectors*. Two curves $\gamma: I \to M$ and $\gamma': I' \to M$ are equivalent if

$$\gamma'(0) = \gamma(0) \tag{1}$$

and

$$\mathrm{D}(f \circ \gamma')(0) = \mathrm{D}(f \circ \gamma)(0) \tag{2}$$

for each differentiable function $f: M \to \mathbb{R}$. The sets I and I' are open neighbourhoods of $0 \in \mathbb{R}$. The symbol D denotes the first derivative of a function. The equivalence class of a curve $\gamma: I \to M$ is denoted by $\mathsf{t}\gamma(0)$. The mapping

$$\begin{aligned} \mathsf{t}\gamma &: \mathbb{R} \to \mathsf{T}M \\ &: s \mapsto \mathsf{t}\gamma(s + \cdot)(0) \end{aligned} \tag{3}$$

is called the *tangent prolongation* of the curve γ. The mapping

$$\begin{aligned} \tau_M &: \mathsf{T}M \to M \\ &: \mathsf{t}\gamma(0) \mapsto \gamma(0) \end{aligned} \tag{4}$$

is called the *tangent bundle projection*.

Let $\eta: M \to N$ be a differentiable mapping. The mapping

$$\mathsf{T}\eta: \mathsf{T}M \to \mathsf{T}N \tag{5}$$

defined by

$$\mathsf{T}\eta(\mathsf{t}\gamma(0)) = \mathsf{t}(\eta \circ \gamma)(0) \tag{6}$$

is called the *tangent mapping* of η.

A vector $v \in \mathsf{T}M$ is said to be *tangent* to a set $C \subset M$ if there is a representative $\gamma: I \to M$ with $\mathrm{im}(\gamma) \subset C$. The set of all vectors tangent to C is denoted by $\mathsf{T}C$.

3. Submanifolds of symplectic manifolds

Let (P, ω) be a symplectic manifold and let $\beta: \mathsf{T}P \to \mathsf{T}^*P$ be the natural isomorphism characterized by

$$\langle w, \beta(v) \rangle = \langle v \wedge w, \omega \rangle \tag{7}$$

for each $v \in \mathsf{T}P$ and each $w \in \mathsf{T}P$ such that $\tau_P(w) = \tau_P(v)$. Let $V \subset \mathsf{T}_pP$ be a vector subspace. The symbol T_pP denotes the set $\tau_p^{-1}(p)$. In general, the fibre $\eta^{-1}(p)$ of a fibration $\eta: Y \to P$ over a point p will be denoted by Y_p. The *polar* V° is the subspace

$$\{q \in \mathsf{T}_p^*P; \forall_{v \in V} \langle v, q \rangle = 0\}. \tag{8}$$

We denote by $V^\P$ the *symplectic polar*

$$\beta^{-1}(V^\circ) = \{w \in \mathsf{T}_pP; \forall_{v \in V} \langle v \wedge w, \omega \rangle = 0\}. \tag{9}$$

If $C \subset P$ is a submanifold, then $\mathsf{T}^\P C$ will denote the set

$$\bigcup_{p \in C} (\mathsf{T}_pC)^\P. \tag{10}$$

We recall that a submanifold $C \subset P$ is said to be *isotropic* if $\mathsf{T}^\P C \supset \mathsf{T}C$. A submanifold $C \subset P$ is said to be *co-isotropic* if $\mathsf{T}^\P C \subset \mathsf{T}C$. The set $\mathsf{T}^\P C$ is called the *characteristic* *distribution* of a co-isotropic submanifold $C \subset P$. The characteristic distribution is Frobenius integrable. Its integral foliation is called the *characteristic foliation* of C. A co-isotropic submanifold of the phase space is called a *first-class constraint set*. A submanifold $C \subset P$ is called a *second-class constraint set* if the symplectic form ω restricted to C is non-degenerate. These terms are consistent with Dirac's terminology. A submanifold $C \subset P$ is said to be *Lagrangian* if $\mathsf{T}^{\P}C = \mathsf{T}C$. For an intrinsic definition of the class of a submanifold of a symplectic manifold see [7].

4. Affine symplectic spaces

An *affine space* is a triple (M, V, μ), where M is a set, V is a real vector space of finite dimension and μ is a mapping $\mu: M \times M \to V$ such that

(1) $\mu(x_3, x_2) + \mu(x_2, x_1) + \mu(x_1, x_3) = 0$;

(2) the mapping $\mu(\cdot, x): M \to V$ is bijective for each $x \in M$.

The set M is said to be an affine space *modelled* on the vector space V.

The tangent bundle $\mathsf{T}M$ of an affine space M is identified with $M \times V$. The cotangent bundle T^*M is identified with $M \times V^*$ and the tangent bundle $\mathsf{T}\mathsf{T}^*M$ is identified with $M \times V^* \times V \times V^*$.

The cotangent bundle T^*M is a symplectic affine space. The canonical symplectic form ω is defined by

$$\langle (x, p, \dot{x}, \dot{p}) \wedge (x, p, \delta x, \delta p), \omega \rangle = \langle \delta x, \dot{p} \rangle - \langle \dot{x}, \delta p \rangle \tag{11}$$

for vectors $(x, p, \dot{x}, \dot{p})$ and $(x, p, \delta x, \delta p)$ in $\mathsf{T}\mathsf{T}^*M$.

5. Implicit differential equations of analytical mechanics

An *implicit first-order differential equation* in a differential manifold P is a subset D of the tangent bundle $\mathsf{T}P$ usually assumed to be a submanifold. A curve $\gamma: I \to P$ is called a *solution* of a differential equation $D \subset \mathsf{T}P$ if $\mathsf{t}\gamma(s) \in D$ for each $s \in I \subset \mathbb{R}$.

The phase space of a mechanical system is a symplectic manifold (P, ω). Implicit differential equations of analytical mechanics are submanifolds of the tangent bundle $\mathsf{T}P$ of the phase space. Three types of differential equations are encountered in modern analytical mechanics.

Let $H: P \to \mathbb{R}$ be a differentiable function on the phase space P. The set

$$D = \{w \in \mathsf{T}P;\ p = \tau_P(w), \forall_{u \in \mathsf{T}_p P} \langle u \wedge w, \omega \rangle = \langle u, \mathrm{d}H \rangle\} \tag{12}$$

is the image of the Hamiltonian vector field

$$X: P \to \mathsf{T}P \tag{13}$$

characterized by

$$\mathrm{i}_X \omega = -\mathrm{d}H. \tag{14}$$

This is the simplest type of a differential equation of analytical mechanics called a *Hamiltonian system*. The image of a vector field is called an *explicit differential equation*. Explicit differential equations are integrable in the sense defined in the next section.

Let $C \subset P$ be a submanifold of the phase space and let $H: C \to \mathbb{R}$ be a differentiable function. The set

$$D = \{w \in \mathsf{T}P;\ p = \tau_P(w) \in C, \forall_{u \in \mathsf{T}_p C} \langle u \wedge w, \omega \rangle = \langle u, \mathrm{d}H \rangle\} \tag{15}$$

is a *generalized Hamiltonian system* [10] known as a *Dirac system*. Truly implicit differential equations of this type appear in gauge independent formulations of the dynamics of charged particles [7].

Let $\eta\colon Y \to P$ be a differential fibration and let $G\colon Y \to \mathbb{R}$ be a differentiable function interpreted as a family of functions defined on fibres of η. Under certain regularity conditions the set

$$D = \{w \in \mathsf{T}P;\, \exists_{y\in Y_{\tau_P(w)}} \forall_{u\in \mathsf{T}_y Y} \langle \mathsf{T}\eta(u) \wedge w, \omega\rangle = \langle u, \mathrm{d}G\rangle\} \tag{16}$$

is a submanifold of $\mathsf{T}P$ of dimension equal to the dimension of P. In this case the function G is called a *Morse family of functions on fibres of* η and is represented by the diagram

$$\begin{array}{ccc} Y & \xrightarrow{G} & \mathbb{R} \\ {\scriptstyle\eta}\big\downarrow & & \\ P & & \end{array} \tag{17}$$

The implicit differential equation D will be called a *generalized Dirac system* generated by the Morse family G.

The intrinsic form of the regularity condition is somewhat complicated. We denote by $\mathsf{V}Y$ the *vertical subbundle* defined by

$$\mathsf{V}Y = \{v \in \mathsf{T}Y;\, \mathsf{T}\eta(v) = 0\}. \tag{18}$$

The *critical set* for a family $G\colon Y \to \mathbb{R}$ is the set

$$S(G,\eta) = \{y \in Y;\, \forall_{v\in \mathsf{V}_y Y} \langle v, \mathrm{d}G\rangle = 0\}. \tag{19}$$

At each point $y \in S(G,\eta)$ we define a bilinear mapping

$$\begin{aligned} W(G,y)\colon & \mathsf{V}_y Y \times \mathsf{T}_y Y \to \mathbb{R} \\ &: (v,w) \mapsto \mathrm{D}^{(1,1)}(G\circ\chi)(0,0) \end{aligned} \tag{20}$$

where χ is a mapping from $\mathbb{R}^2$ to Y such that $v = \mathrm{t}\chi(\cdot,0)(0)$ and $w = \mathrm{t}\chi(0,\cdot)(0)$. The symbol $D^{(1,1)}$ denotes the second partial derivative $\partial^2/\partial s\partial t$ of a function of two variables $(s,t) \in \mathbb{R}^2$. The family G is a Morse family if the rank of $W(G,y)$ is maximal at each $y \in S(G,\eta)$. In terms of Darboux coordinates (x^κ, p_λ) in P and adapted coordinates $(x^\kappa, p_\lambda, y^A)$ in Y the mapping $W(G,y)$ is represented by the matrix

$$\left[\frac{\partial^2 G}{\partial y^A \partial y^B}, \frac{\partial^2 G}{\partial y^A \partial x^\kappa}, \frac{\partial^2 G}{\partial y^A \partial p_\lambda}\right]. \tag{21}$$

The rank of this matrix must be maximal at points of the critical set $S(G,\eta)$ [11]. The coordinates $(x^\kappa, p_\lambda, \dot{x}^\mu, \dot{p}_\nu)$ of an element of the generalized Dirac system (16) satisfy equations

$$\dot{x}^\mu = \frac{\partial G}{\partial p_\mu} \qquad \dot{p}_\nu = -\frac{\partial G}{\partial x^\nu} \qquad \frac{\partial G}{\partial y^A} = 0 \tag{22}$$

for some values of the coordinates (y^A).

Example 1. Let $P = \mathbb{R}^4$ and let $\mathsf{T}P$ be identified with $\mathbb{R}^8$. Let

$$\begin{aligned} G\colon & P \times \mathbb{R}^2 \to \mathbb{R} \\ &: (x,y,p,q,\lambda,\mu) \mapsto \lambda^4 - 6\lambda^2 - \lambda x + \mu q + y^2 + p \end{aligned} \tag{23}$$

and let $\eta: P \times \mathbb{R}^2 \to P$ be the canonical projection. The family

$$\begin{array}{ccc} P \times \mathbb{R}^2 & \xrightarrow{G} & \mathbb{R} \\ \eta \downarrow & & \\ P & & \end{array} \tag{24}$$

is a Morse family. The critical set for G is the set

$$S(G, \eta) = \{(x, y, p, q, \lambda, \mu) \in P \times \mathbb{R}^2; 4\lambda^3 - 12\lambda - x = 0, q = 0\}. \tag{25}$$

The mapping $W(G, (x, y, p, q, \lambda, \mu))$ is represented by the matrix

$$\begin{bmatrix} \frac{\partial^2 G}{\partial\lambda^2} & \frac{\partial^2 G}{\partial\lambda\partial\mu} & \frac{\partial^2 G}{\partial\lambda\partial x} & \frac{\partial^2 G}{\partial\lambda\partial y} & \frac{\partial^2 G}{\partial\lambda\partial p} & \frac{\partial^2 G}{\partial\lambda\partial q} \\ \frac{\partial^2 G}{\partial\mu\partial\lambda} & \frac{\partial^2 G}{\partial\mu^2} & \frac{\partial^2 G}{\partial\mu\partial x} & \frac{\partial^2 G}{\partial\mu\partial y} & \frac{\partial^2 G}{\partial\mu\partial p} & \frac{\partial^2 G}{\partial\mu\partial q} \end{bmatrix} = \begin{bmatrix} 12\lambda^2 - 12 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{26}$$

This matrix is of rank 2 at each point $(x, y, p, q, \lambda, \mu) \in S(G, \eta)$.

The generalized Dirac system generated by G is the implicit differential equation

$$D = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P; q = 0, \dot{x} - 1 = 0, 4\dot{p}^3 - 12\dot{p} - x = 0, \dot{q} + 2y = 0\}. \tag{27}$$

Example 2. Let $P = \mathbb{R}^2$ and let $\mathsf{T}P$ be identified with $\mathbb{R}^4$. The family of functions

$$\begin{aligned} G&: P \times \mathbb{R} \to \mathbb{R} \\ &: (x, p, \lambda) \mapsto \lambda^3 - 3\lambda^2 - \lambda x - p^2 \end{aligned} \tag{28}$$

defined on fibres of the canonical projection $\eta: P \times \mathbb{R} \to P$ is a Morse family. The critical set for G is the set

$$S(G, \eta) = \{(x, p, \lambda) \in P \times \mathbb{R}; 3\lambda^2 - 6\lambda - x = 0\}. \tag{29}$$

The mapping $W(G, (x, p, \lambda))$ is represented by the matrix

$$\left[\frac{\partial^2 G}{\partial\lambda^2}, \quad \frac{\partial^2 G}{\partial\lambda\partial x}, \quad \frac{\partial^2 G}{\partial\lambda\partial p} \right] = [6\lambda - 6, -1, 0]. \tag{30}$$

This matrix is of rank 1 at each point $(x, p, \lambda) \in S(G, \eta)$.

The family G generates the generalized Dirac system

$$D = \{(x, p, \dot{x}, \dot{p}) \in \mathsf{T}P; \dot{x} + 2p = 0, 3\dot{p}^2 - 6\dot{p} - x = 0\}. \tag{31}$$

It is known that the tangent bundle $\mathsf{T}P$ together with the derived 2-form $\mathrm{d}_T\omega$ form a symplectic manifold $(\mathsf{T}P, \mathrm{d}_T\omega)$ [9, 10]. All types of implicit differential equations of analytical mechanics are Lagrangian submanifolds of this symplectic manifold.

Most Dirac systems are also generalized Dirac systems. Let W be a vector space and let $K: P \to W$ be a differentiable mapping such that $C = K^{-1}(0)$ is a submanifold of P. Let $\tilde{H}: P \to \mathbb{R}$ be a differentiable function. The Dirac system (15) generated by the function $H = \tilde{H}|C$ is a generalized Dirac system generated by the Morse family

$$\begin{array}{ccc} P \times W^* & \xrightarrow{G} & \mathbb{R} \\ \eta \downarrow & & \\ P & & \end{array} \tag{32}$$

where G is the function

$$\begin{aligned} G&: P \times W^* \to \mathbb{R} \\ &: (p, \lambda) \mapsto \tilde{H}(p) + \langle K(p), \lambda \rangle \end{aligned} \tag{33}$$

and

$$\eta: P \times W^* \to P$$
$$: (p, \lambda) \mapsto p \tag{34}$$

is the canonical projection. The representation of Dirac systems as generalized Dirac systems provides an unification of the analysis of integrability. The constraint set C may not be representable globally in the form $C = K^{-1}(0)$. In this case the Dirac system (15) is still a union of generalized Dirac systems.

6. Integrability of implicit differential equations

An implicit differential equation $D \subset \mathsf{T}P$ is said to be *integrable* if for each $v \in D$ there is a solution $\gamma: I \to P$ such that $\mathsf{t}\gamma(0) = v$.

Proposition 1. If $D \subset \mathsf{T}P$ is integrable, then

$$D \subset \mathsf{T}(\tau_P(D)). \tag{35}$$

Proof. Let $v \in D$ and let $\gamma: I \to P$ be a solution of D such that $\mathsf{t}\gamma(0) = v$. For each $s \in I$ we have $\mathsf{t}\gamma(s) \in D$. It follows that $\gamma(s) \in \tau_P(D)$ for each $s \in I$. Consequently $\mathsf{t}\gamma(s) \in \mathsf{T}(\tau_P(D))$ for each $s \in I$ and $v = \mathsf{t}\gamma(0) \in \mathsf{T}(\tau_P(D))$. □

Proposition 2. If $C = \tau_P(D)$ is a submanifold of P and if the mapping

$$\tau: D \to C$$
$$: v \mapsto \tau_P(v) \tag{36}$$

is a surjective submersion, then the condition $D \subset \mathsf{T}C$ is sufficient for integrability of the implicit differential equation $D \subset \mathsf{T}P$.

Proof. Let v be an element of D and let $p = \tau_P(v)$. Let $\sigma: C \to D$ be a (local) section of $\tau: D \to C$ such that $\sigma(p) = v$. If $\varepsilon: D \to \mathsf{T}C$ is the canonical injection, then $X = \varepsilon \circ \sigma: C \to \mathsf{T}C$ is a section of $\tau_C: \mathsf{T}C \to C$, hence a vector field on C. Let $\gamma: I \to C$ be an integral curve of X such that $\gamma(0) = p$. Then $\mathrm{im}(\mathsf{t}\gamma) \subset D$ and $\mathsf{t}\gamma(0) = X(p) = v$. □

An implicit differential equation $D \subset \mathsf{T}P$ is said to be *integrable* at $v \in D$ if there is a solution $\gamma: I \to P$ of D such that $\mathsf{t}\gamma(0) = v$. The set

$$\tilde{D} = \{v \in D;\ D \text{ is integrable at } v\} \tag{37}$$

is called the *integrable part* of D. The implicit equation D is *integrable* if and only if $D = \tilde{D}$.

Proposition 3. The integrable part $\tilde{D}$ of a differential equation $D \subset \mathsf{T}P$ is an integrable differential equation.

Proof. Let $v \in \tilde{D}$ and let $\gamma: I \to P$ be a solution of D such that $\mathsf{t}\gamma(0) = v$. For each $s \in I$ the curve

$$\gamma(\cdot + s): I - s \to P$$
$$: t \mapsto \gamma(t + s) \tag{38}$$

is a solution of D. Hence, $\mathsf{t}\gamma(s) \in \tilde{D}$. It follows that $\tilde{D}$ is integrable. □

We have the obvious relations

$$\tilde{D} \subset \mathsf{T}(\tau_P(\tilde{D})) \subset \mathsf{T}(\tau_P(D)) \tag{39}$$

and

$$\tilde{D} \subset D \cap \mathsf{T}(\tau_P(D)). \tag{40}$$

Proposition 4. If $\tilde{D}$ is the integrable part of a differential equation $D \subset \mathsf{T}P$ and D' is a subset of D such that $\tilde{D} \subset D'$, then $\tilde{D}$ is the integrable part of D'.

Proof. If $D' \subset D$ and D' is integrable at v, then $v \in \tilde{D}$ since a solution of D' is a solution of D. Hence, $\tilde{D}' \subset \tilde{D}$. If $\tilde{D} \subset D'$, then D' is integrable at each $v \in \tilde{D}$. Hence, $\tilde{D} \subset \tilde{D}'$.□

Propositions 2 and 4 suggest a simple algorithm for extracting the integrable part of a sufficiently regular implicit differential equation without actually solving the equation.

Let $D \subset \mathsf{T}P$ be an implicit differential equation. We consider the sequence of sets

$$C^0 = \tau_P(D), C^1 = \tau_P(D \cap \mathsf{T}C^0), \ldots, C^k = \tau_P(D \cap \mathsf{T}C^{k-1}), \ldots \tag{41}$$

and the sequence of differential equations

$$D^0 = D, D^1 = D \cap \mathsf{T}C^0, \ldots, D^k = D \cap \mathsf{T}C^{k-1}, \ldots. \tag{42}$$

It follows from $\tau_P(\mathsf{T}C^k) = C^k$ that $C^{k+1} = \tau_P(D \cap \mathsf{T}C^k) \subset C^k$ for each $k \geqslant 0$. Consequently, $D^{k+1} \subset D^k$ for each $k \geqslant 0$. It is easily seen that the integrable part $\tilde{D}$ of D is the integrable part of all equations in the sequence (42). It may happen that after a finite number of steps the sets in the sequence (41) are all equal to a set $\overline{C}$. This set satisfies the equality

$$\overline{C} = \tau_P(D \cap \mathsf{T}\overline{C}). \tag{43}$$

If the differential equation $\overline{D} = D \cap \mathsf{T}\overline{C}$ is integrable, then it is the integrable part of D.

The following proposition implies that for a sufficiently regular differential equation in a finite-dimensional space P the sets in the sequence (41) after a finite number of steps are all equal to a set $\overline{C}$.

Proposition 5. If C^k and C^{k-1} are submanifolds of P of the same dimension, then $C^l = C^k$ for each $l \geqslant k$.

Proof. If $p \in C^k$, then the set $D \cap \mathsf{T}_pC^{k-1}$ is not empty. If C^k has the same dimension as C^{k-1}, then $\mathsf{T}_pC^k = \mathsf{T}_pC^{k-1}$ since C^k is an open submanifold of C^{k-1}. Consequently the set $D \cap \mathsf{T}_pC^k$ is not empty and $p \in C^{k+1}$. This implies that $C^{k+1} = C^k$. Hence, $C^l = C^k$ for each $l \geqslant k$. □

If the equation D is sufficiently regular, then the equation $\overline{D}$ is integrable since $\overline{C}$ is a submanifold of P and the mapping

$$\begin{aligned} \bar{\tau}&: D \cap \mathsf{T}\overline{C} \to \overline{C} \\ &: v \mapsto \tau_P(v) \end{aligned} \tag{44}$$

is a surjective submersion.

The present integrability algorithm is a modification of the algorithm described in [6].

Example 3. The implicit differential equation

$$D = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\ q = 0, \dot{x} - 1 = 0, 4\dot{p}^3 - 12\dot{p} - x = 0, \dot{q} + 2y = 0\} \tag{45}$$

of example 1 is not integrable. The extraction algorithm sequence

$$C^0 = \{(x, y, p, q) \in P;\ q = 0\} \tag{46}$$
$$\mathsf{T}C^0 = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\ q = 0, \dot{q} = 0\} \tag{47}$$
$$D \cap \mathsf{T}C^0 = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\ y = 0, q = 0, \dot{x} - 1 = 0, 4\dot{p}^3 - 12\dot{p} - x = 0, \dot{q} = 0\} \tag{48}$$
$$C^1 = \{(x, y, p, q) \in P;\ y = 0, q = 0\} \tag{49}$$
$$\mathsf{T}C^1 = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\ y = 0, q = 0, \dot{y} = 0, \dot{q} = 0\} \tag{50}$$
$$D \cap \mathsf{T}C^1 = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\ y = 0, q = 0, \dot{x} - 1 = 0, 4\dot{p}^3 - 12\dot{p} - x = 0, \dot{y} = 0, \dot{q} = 0\} \tag{51}$$
$$C^2 = C^1 \tag{52}$$

terminates with $\overline{C} = C^1$ and $\overline{D} = D \cap \mathsf{T}C^1$. The differential equation $\overline{D}$ is not the integrable part of D since it is not integrable. The criterion of proposition 2 in [6] is not satisfied. An improved version of the algorithm is suggested by this proposition. Let D be represented as the union

$$D = D_+ \cup D_- \tag{53}$$

of

$$D_+ = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\ \dot{x} - 1 = 0, 4\dot{p}^3 - 12\dot{p} - x = 0, q = 0, \dot{q} + 2y = 0, \dot{p} > -1/2\} \tag{54}$$

and

$$D_- = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\ \dot{x} - 1 = 0, 4\dot{p}^3 - 12\dot{p} - x = 0, q = 0, \dot{q} + 2y = 0, \dot{p} < 1/2\}. \tag{55}$$

The extraction algorithm applied to the component D_+ produces the sequence

$$C^0_+ = \{(x, y, p, q) \in P;\ x \geqslant -8, q = 0\} \tag{56}$$
$$\mathsf{T}C^0_+ = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\ x \geqslant -8, q = 0, \dot{q} = 0, \dot{x} = 0 \text{ if } x = -8\} \tag{57}$$
$$D_+ \cap \mathsf{T}C^0_+ = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\ x > -8, y = 0, q = 0, \dot{x} - 1 = 0, \dot{p} > -1/2, 4\dot{p}^3 - 12\dot{p} - x = 0, \dot{q} = 0\} \tag{58}$$
$$C^1_+ = \{(x, y, p, q) \in P;\ x > -8, y = 0, q = 0\} \tag{59}$$
$$\mathsf{T}C^1_+ = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\ x > -8, y = 0, q = 0, \dot{y} = 0, \dot{q} = 0\} \tag{60}$$
$$D_+ \cap \mathsf{T}C^1_+ = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\ x > -8, y = 0, q = 0, \dot{x} - 1 = 0, \dot{p} > -1/2, 4\dot{p}^3 - 12\dot{p} - x = 0, \dot{y} = 0, \dot{q} = 0\} \tag{61}$$
$$C^2_+ = C^1_+ \tag{62}$$

terminating with $\overline{C_+} = C^1_+$ and $\overline{D_+} = D_+ \cap \mathsf{T}C^1_+$. For D_- we have

$$\overline{C_-} = \{(x, y, p, q) \in P;\ x < 8, y = 0, q = 0\} \tag{63}$$

and

$$\overline{D_-} = \{(x, y, p, q, \dot{x}, \dot{y}, \dot{p}, \dot{q}) \in \mathsf{T}P;\, x < 8,\, y = 0,\, q = 0,\, \dot{x} - 1 = 0,\, \dot{p} < 1/2,\\ 4\dot{p}^3 - 12\dot{p} - x = 0,\, \dot{y} = 0,\, \dot{q} = 0\}. \tag{64}$$

Equations $\overline{D_+}$ and $\overline{D_-}$ are integrable. The union $\overline{D_+} \cup \overline{D_-}$ is the integrable part $\tilde{D}$ of D.

Dirac represented his generalized Hamiltonian systems as affine spaces of Hamiltonian vector fields restricted to constraint sets. This description does not apply to the differential equation $\tilde{D} = \overline{D_+} \cup \overline{D_-}$ obtained by the improved algorithm. It is also clear that in this example the integrability algorithm is applied to the differential equation itself and not to its constraint set. The frequently used term 'constraint algorithm' is in our opinion misleading.

Example 4. A different anomaly of an implicit differential equation is present in the implicit differential equation

$$D = \{(x, p, \dot{x}, \dot{p}) \in \mathsf{T}P;\, \dot{x} + 2p = 0,\, 3\dot{p}^2 - 6\dot{p} - x = 0\} \tag{65}$$

of example 2. This equation is not integrable. The basic extraction algorithm produces the sequence

$$C^0 = \{(x, p) \in P;\, x \geqslant -3\} \tag{66}$$

$$\mathsf{T}C^0 = \{(x, p, \dot{x}, \dot{p}) \in \mathsf{T}P;\, x \geqslant -3,\, \dot{x} = 0 \text{ if } x = -3\} \tag{67}$$

$$D \cap \mathsf{T}C^0 = \{(x, p, \dot{x}, \dot{p}) \in \mathsf{T}P;\, \dot{x} = 0 \text{ if } x = -3,\, \dot{x} + 2p = 0,\, 3\dot{p}^2 - 6\dot{p} - x = 0\} \tag{68}$$

$$C^1 = \{(x, p) \in P;\, x \geqslant -3,\, p = 0 \text{ if } x = -3\} \tag{69}$$

$$\mathsf{T}C^1 = \{(x, p, \dot{x}, \dot{p}) \in \mathsf{T}P;\, x \geqslant -3,\, p = 0 \text{ and } \dot{x} = 0 \text{ if } x = -3\} \tag{70}$$

$$D \cap \mathsf{T}C^1 = D \cap \mathsf{T}C^0 \tag{71}$$

terminating with $\overline{C} = C^1$ and $\overline{D} = D \cap \mathsf{T}C^0$. The differential equation $\overline{D}$ is not integrable at the point $(x, p, \dot{x}, \dot{p}) = (-3, 0, 0, 1)$. The necessary integrability condition of proposition 4 in [6] is not met. The following operation is suggested by this proposition. The integrable part $\tilde{D}$ of D is obtained as the projection

$$\begin{aligned}\tilde{D} &= \tau_{\mathsf{T}P}(\tau_{\mathsf{TT}P}(\mathsf{TT}\overline{D} \cap \mathsf{T}^3 P))\\ &= \{(x, p, \dot{x}, \dot{p}) \in \mathsf{T}P;\, x > -3,\, \dot{x} + 2p = 0,\, 3\dot{p}^2 - 6\dot{p} - x = 0\}\end{aligned} \tag{72}$$

of the formal prolongation

$$\mathsf{TT}\overline{D} \cap \mathsf{T}^3 P = \{(x, p, \dot{x}, \dot{p}, \ddot{x}, \ddot{p}, \dddot{x}, \dddot{p}) \in \mathsf{T}^3 P;\, \dot{x} = 0 \text{ if } x = -3,\, \dot{x} + 2p = 0,\\ 3\dot{p}^2 - 6\dot{p} - x = 0,\, \ddot{x} + 2\dot{p} = 0,\, 6\dot{p}\ddot{p} - 6\ddot{p} - \dot{x} = 0,\, \dddot{x} + 2\ddot{p} = 0,\\ 6\ddot{p}^2 + 6\dot{p}\ddot{p} - 6\dddot{p} - \ddot{x} = 0\} \tag{73}$$

of $\overline{D}$.

Implicit differential equations in the above examples are generalized Dirac systems. The initial analysis of their integrability can be performed in terms of their generating Morse families using an adaptation of the extraction algorithm described in section 8. We have not adapted the modified algorithms used in these examples to generalized Dirac systems since we do not know examples of anomalous systems with physical interpretations.

7. Integrability of Dirac systems

Let (P, ω) be a symplectic phase space of a mechanical system and let

$$D = \{w \in \mathsf{T}P;\ p = \tau_P(w) \in C, \forall_{u \in \mathsf{T}_p C} \langle u \wedge w, \omega\rangle = \langle u, \mathrm{d}H\rangle\} \tag{74}$$

be a Dirac system generated by a Hamiltonian $H: C \rightarrow \mathbb{R}$ defined on a submanifold $C \subset P$. The set

$$D_p = \{w \in \mathsf{T}_p P; \forall_{u \in \mathsf{T}_p C} \langle u \wedge w, \omega\rangle = \langle u, \mathrm{d}H\rangle\} \tag{75}$$

is an affine subspace of $\mathsf{T}_p P$ modelled on the vector subspace

$$\mathsf{T}_p^{\P} C = \{w \in \mathsf{T}_p P; \forall_{u \in \mathsf{T}_p C} \langle u \wedge w, \omega\rangle = 0\}. \tag{76}$$

The image $\tau_P(D)$ is the submanifold C. The mapping τ defined in formula (36) is a submersion since local sections of this mapping are easily constructed. It follows that the condition $D \subset \mathsf{T}C$ is sufficient for integrability of the Dirac system D.

For the space $\mathsf{T}_p C$ we use the representation

$$\begin{aligned} \mathsf{T}_p C &= (\mathsf{T}_p^{\P} C)^{\P} \\ &= \{w \in \mathsf{T}_p P; \forall_{u \in \mathsf{T}_p^{\P} C} \langle u \wedge w, \omega\rangle = 0\}. \end{aligned} \tag{77}$$

We denote by $\ker(\mathrm{d}H(p))$ the space

$$\{u \in \mathsf{T}_p P; u \in \mathsf{T}_p C, \langle u, \mathrm{d}H\rangle = 0\}. \tag{78}$$

Lemma 1. If $\mathsf{T}_p^{\P} C \subset \ker(\mathrm{d}H(p))$, then $D_p \subset \mathsf{T}_p C$.

Proof. Let $w \in D_p$. From the definition (75) we have $\langle u \wedge w, \omega\rangle = \langle u, \mathrm{d}H\rangle$ for each $u \in \mathsf{T}_p C$. If $\mathsf{T}_p^{\P} C \subset \ker(\mathrm{d}H(p))$, then $\langle u \wedge w, \omega\rangle = 0$ for each $u \in \mathsf{T}_p^{\P} C$. Hence, $w \in (\mathsf{T}_p^{\P} C)^{\P} = \mathsf{T}_p C$. □

Lemma 2. If $D_p \subset \mathsf{T}_p C$, then $\mathsf{T}_p^{\P} C \subset \mathsf{T}_p C$.

Proof. Let $v \in \mathsf{T}_p^{\P} C$. If $w \in D_p$, then $w + v \in D_p$ since D_p is an affine space modelled on $\mathsf{T}_p^{\P} C$. If $D_p \subset \mathsf{T}_p C$, then $w \in \mathsf{T}_p C$ and $w + v \in \mathsf{T}_p C$. Hence, $v \in \mathsf{T}_p C$. □

Lemma 3. If $D_p \subset \mathsf{T}_p C$, then $\mathsf{T}_p^{\P} C \subset \ker(\mathrm{d}H(p))$.

Proof. If $D_p \subset \mathsf{T}_p C$, then $D_p \cap \mathsf{T}_p C$ is not empty. If $w \in D_p \cap \mathsf{T}_p C$ and $u \in \mathsf{T}_p C \cap \mathsf{T}_p^{\P} C$, then $\langle u, \mathrm{d}H\rangle = \langle u \wedge w, \omega\rangle = 0$ since $u \in \mathsf{T}_p C$ and $w \in \mathsf{T}_p^{\P} C$. It follows that $u \in \ker(\mathrm{d}H(p))$.
□

Theorem 1. A Dirac system

$$D = \{w \in \mathsf{T}P;\ p = \tau_P(w) \in C, \forall_{u \in \mathsf{T}_p C} \langle u \wedge w, \omega\rangle = \langle u, \mathrm{d}H\rangle\} \tag{79}$$

is integrable if and only if the submanifold $C \subset P$ is co-isotropic and the Hamiltonian function $H: C \rightarrow \mathbb{R}$ is constant on leaves of the characteristic foliation of C.

Proof.

(1) If C is co-isotropic and H is constant on leaves of the characteristic foliation of C, then $\mathsf{T}_p^{\P} C \subset \ker(\mathrm{d}H(p))$ for each $p \in C$. By lemma 1 this implies $D_p \subset \mathsf{T}_p C$ for each $p \in C$. Hence, D is integrable.

(2) If D is integrable, then $D_p \subset \mathsf{T}_p C$ for each $p \in C$. By lemma 2 and lemma 3 this implies that C is co-isotropic and H is constant on leaves of the characteristic foliation of C. □

If the Dirac system D is not integrable, then the algorithm for extracting its integrable part can be tried. An adaptation of this algorithm to Dirac systems is described in [6].

8. Integrability of generalized Dirac systems

Let

$$\begin{tikzcd} Y \arrow[r, "G"] \arrow[d, "\eta"'] & \mathbb{R} \\ P & \end{tikzcd} \tag{80}$$

be a Morse family generating the implicit differential equation

$$D = \{w \in \mathsf{T}P;\ \exists_{y \in Y_{\tau_P(w)}} \forall_{u \in \mathsf{T}_y Y} \langle \mathsf{T}\eta(u) \wedge w, \omega \rangle = \langle u, \mathrm{d}G \rangle\}. \tag{81}$$

The set

$$S = \{y \in Y;\ \forall_{u \in \mathsf{T}_y Y} \mathsf{T}\eta(u) = 0 \Rightarrow \langle u, \mathrm{d}G \rangle = 0\} \tag{82}$$

is called the *critical set* and the set

$$C = \{p \in P;\ \exists_{y \in Y_p} \forall_{u \in \mathsf{T}_y Y} \mathsf{T}\eta(u) = 0 \Rightarrow \langle u, \mathrm{d}G \rangle = 0\} \tag{83}$$

is called the *constraint set*. The sets D, S and C are related by

$$C = \tau_P(D) = \eta(S). \tag{84}$$

The algorithm described in section 5 produces a sequence of sets

$$C^0, C^1, \ldots, C^k, \ldots, \tag{85}$$

where

$$C^0 = C \tag{86}$$

and

$$C^k = \{p \in P;\ p \in C^{k-1} \text{ and } \exists_{y \in Y_p} \forall_{u \in \mathsf{T}_y Y} \mathsf{T}\eta(u) \in \mathsf{T}^{\P}_p C^{k-1} \Rightarrow \langle u, \mathrm{d}G \rangle = 0\} \tag{87}$$

for $k > 0$. The terminal constraint set $\overline{C}$ is the set

$$\overline{C} = \{p \in P;\ \exists_{y \in Y_p} \forall_{u \in \mathsf{T}_y Y} \mathsf{T}\eta(u) \in \mathsf{T}^{\P}_p \overline{C} \Rightarrow \langle u, \mathrm{d}G \rangle = 0\}. \tag{88}$$

The terminal critical set $\overline{S}$ and the terminal equation $\overline{D}$ are the sets

$$\overline{S} = \{y \in Y;\ p = \eta(y) \in \overline{C} \text{ and } \exists_{y \in Y_p} \forall_{u \in \mathsf{T}_y Y} \mathsf{T}\eta(u) \in \mathsf{T}^{\P}_p \overline{C} \Rightarrow \langle u, \mathrm{d}G \rangle = 0\} \tag{89}$$

and

$$\overline{D} = \{w \in \mathsf{T}P;\ p = \tau_P(w) \in \overline{C} \text{ and } \exists_{y \in \overline{S}_p} \forall_{u \in \mathsf{T}_y Y} \langle \mathsf{T}\eta(u) \wedge w, \omega \rangle = \langle u, \mathrm{d}G \rangle\}. \tag{90}$$

We have the relation

$$\overline{D} = D \cap \mathsf{T}\overline{C}. \tag{91}$$

This version of the extraction algorithm can be applied to a Dirac system represented by the Morse family (32). This is an alternative to the direct application of the procedure used in [6]. If $C^1 = C^0$, then the Dirac system is integrable.

9. Examples

Example 5. It is possible to give the dynamics of a non-relativistic particle of mass m and charge e a gauge independent form in a four-dimensional configuration space analogous to the five-dimensional space of Kaluza. The dynamics is an integrable Dirac system described in [7]. This is the only true finite-dimensional Dirac system related to physics known to us.

Example 6. Let V be an Euclidean vector space with a metric tensor $g: V \to V^*$. The implicit differential equation [4]

$$\begin{aligned} D = \{(x, p, \dot{x}, \dot{p}) \in V \times V^* \times V \times V^*; \; x \neq 0, \; p = k(\|x\|^2 g(\dot{x}) - \langle \dot{x}, g(x)\rangle g(x)) \\ \dot{p} = k(\|\dot{x}\|^2 g(x) - \langle \dot{x}, g(x)\rangle g(\dot{x}))\} \end{aligned} \tag{92}$$

is a Dirac system generated by the Hamiltonian

$$\begin{aligned} H&: C \to \mathbb{R} \\ &: (x, p) \mapsto \frac{\|p\|^2}{2k\|x\|^2} \end{aligned} \tag{93}$$

defined on the constraint set

$$C = \{(x, p) \in V \times V^*; \; x \neq 0, \; \langle x, p\rangle = 0\}. \tag{94}$$

This system can be presented as a generalized Dirac system generated by the Morse family

$$\begin{array}{ccc} \mathring{V} \times V^* \times \mathbb{R} & \xrightarrow{G} & \mathbb{R} \\ \eta \big\downarrow & & \\ \mathring{V} \times V^* & & \end{array} \tag{95}$$

where

$$\mathring{V} = \{x \in V; \; x \neq 0\} \tag{96}$$

and G is the function

$$\begin{aligned} G&: \mathring{V} \times V^* \times \mathbb{R} \to \mathbb{R} \\ &: (x, p, \lambda) \mapsto \frac{\|p\|^2}{2k\|x\|^2} + \lambda\langle x, p\rangle \end{aligned} \tag{97}$$

and

$$\begin{aligned} \eta&: \mathring{V} \times V^* \times \mathbb{R} \to \mathring{V} \times V^* \\ &: (x, p, \lambda) \mapsto (x, p) \end{aligned} \tag{98}$$

is the canonical projection.

The system is not integrable. The integrable part $\overline{D}$ is obtained through the following steps:

$$C^0 = \{(x, p) \in \mathring{V} \times V^*; \; \langle x, p\rangle = 0\} \tag{99}$$

$$\mathsf{T}C^0 = \{(x, p, \delta x, \delta p) \in \mathring{V} \times V^* \times V \times V^*; \; \langle x, p\rangle = 0, \; \langle \delta x, p\rangle + \langle x, \delta p\rangle = 0\} \tag{100}$$

$$\begin{aligned} \mathsf{T}^\P C^0 = \Bigg\{(x, p, \delta x, \delta p) \in \mathring{V} \times V^* \times V \times V^*; \; \langle x, p\rangle = 0, \; \delta x = \frac{\langle \delta x, g(x)\rangle}{\|x\|^2} x, \\ \delta p = -\frac{\langle \delta x, g(x)\rangle}{\|x\|^2} p\Bigg\} \end{aligned} \tag{101}$$

$$C^1 = \{(x, p) \in \mathring{V} \times V^*; \; p = 0\} \tag{102}$$

$$\mathsf{T}C^1 = \{(x, p, \delta x, \delta p) \in \mathring{V} \times V^* \times V \times V^*;\ p = 0, \delta p = 0\} \tag{103}$$
$$\mathsf{T}^\P C^1 = \mathsf{T}C^1 \tag{104}$$
$$C^2 = C^1 = \overline{C} \tag{105}$$

and

$$\overline{D} = \left\{ (x, p, \dot{x}, \dot{p}) \in \mathring{V} \times V^* \times V \times V^*;\ p = 0, \dot{x} = \frac{\langle \dot{x}, g(x)\rangle}{\|x\|^2}\, x, \dot{p} = 0 \right\}. \tag{106}$$

Solutions are curves of the form

$$\begin{aligned} \gamma &: \mathbb{R} \to \mathring{V} \times V^* \\ &: t \mapsto (f(t)u, 0) \end{aligned} \tag{107}$$

where $u \in \mathring{V}$ and f is a function such that $f(t) \neq 0$.

Example 7. Let (M, V, μ) be the affine spacetime of special relativity with the Minkowski metric $g \colon V \to V^*$. Dynamics of a free particle of mass m is the generalized Dirac system

$$\begin{aligned} D &= \left\{ (x, p, \dot{x}, \dot{p}) \in M \times V^* \times V \times V^*;\ \langle g^{-1}(p), p\rangle > 0, \dot{p} = 0, \exists_{\lambda \in \mathbb{R}_+} \dot{x} = \frac{\lambda}{m} g^{-1}(p) \right\} \\ &= \left\{ (x, p, \dot{x}, \dot{p}) \in M \times V^* \times V \times V^*;\ \langle \dot{x}, g(\dot{x})\rangle > 0, p = \frac{m}{\|\dot{x}\|} g(\dot{x}), \dot{p} = 0 \right\} \end{aligned} \tag{108}$$

generated by the Morse family

$$\begin{array}{ccc} M \times K_* \times \mathbb{R}_+ & \xrightarrow{G} & \mathbb{R} \\ \Big\downarrow{\scriptstyle \eta} & & \\ M \times K_* & & \end{array} \tag{109}$$

where

$$K_* = \{p \in V^*;\ \langle g^{-1}(p), p\rangle > 0\} \tag{110}$$

$$\begin{aligned} \eta &: M \times K_* \times \mathbb{R}_+ \to M \times K_* \\ &: (x, p, \lambda) \mapsto (x, p) \end{aligned} \tag{111}$$

is the canonical projection and

$$\begin{aligned} G &: M \times K_* \times \mathbb{R}_+ \to \mathbb{R} \\ &: (x, p, \lambda) \mapsto \lambda\,(\|p\| - m)\,. \end{aligned} \tag{112}$$

The set

$$S = \{(x, p, \lambda) \in M \times K_* \times \mathbb{R}_+;\ \|p\| = m\} \tag{113}$$

is the critical set for the Morse family and the mass shell

$$C = \{(x, p) \in M \times K_*;\ \|p\| = m\} \tag{114}$$

is the constraint set. We have

$$C^0 = C \tag{115}$$
$$\mathsf{T}C^0 = \{(x, p, \dot{x}, \dot{p}) \in M \times K_* \times V \times V^*;\ \|p\| = m, \langle g^{-1}(p), \dot{p}\rangle = 0\} \tag{116}$$
$$\mathsf{T}^\P C^0 = \left\{ (x, p, \delta x, \delta p) \in M \times K_* \times V \times V^*;\ \|p\| = m, \delta x = \frac{\langle \delta x, p\rangle}{m^2}\, p, \delta p = 0 \right\} \tag{117}$$

and

$$C^1 = C^0. \tag{118}$$

It follows that this generalized Dirac system is integrable. Solutions are oriented lines in the affine phase space $M \times V^*$. Their orientation reflects the distinction between particles and antiparticles introduced by Stueckelberg [8] and used by Feynman [3].

The present example shows that even the simplest dynamical systems of relativistic mechanics are generalized Dirac systems. The equation D is derived from the Lagrangian

$$\begin{aligned} L\colon & M \times K \to \mathbb{R} \\ & : (x, \dot{x}) \mapsto m\|\dot{x}\| \end{aligned} \tag{119}$$

where

$$K = \{v \in V; \langle v, g(v)\rangle > 0\}. \tag{120}$$

Since the Lagrangian is homogeneous the system would be considered by some physicists as a Dirac system D' with zero Hamiltonian defined on the mass shell C. This Dirac system provides an incomplete description of relativistic dynamics—the Stueckelberg distinction between particles and antiparticles is lost. The mass shell is co-isotropic since it is a submanifold of codimension one. It follows from theorem 1 that the Dirac system D' is integrable. An alternative proof of integrability of D is obtained by observing that D is an open subset of D'.

Example 8. We analyse the dynamics of two interacting relativistic particles formulated in [12, 13]. Masses of the particles are denoted by m_1 and m_2. The interaction potential is a function U of a real positive argument.

In addition to the notation of the preceding example we introduce symbols

$$N = \{(x_1, x_2) \in M \times M; \langle x_2 - x_1, g(x_2 - x_1)\rangle < 0\} \tag{121}$$

$$P = N \times K_* \times K_* \tag{122}$$

$$\|x_2 - x_1\| = \sqrt{-\langle x_2 - x_1, g(x_2 - x_1)\rangle} \tag{123}$$

for $(x_1, x_2) \in N$,

$$\overline{m}_1 = \sqrt{m_1^2 + U(\|x_2 - x_1\|)} \tag{124}$$

and

$$\overline{m}_2 = \sqrt{m_2^2 + U(\|x_2 - x_1\|)}. \tag{125}$$

Dynamics of the particles is a generalized Dirac system

$$D \subset M \times M \times V^* \times V^* \times V \times V \times V^* \times V^* \tag{126}$$

generated by the Morse family

$$\begin{tikzcd} P \times \mathbb{R}_+ \times \mathbb{R}_+ \arrow[r, "G"] \arrow[d, "\eta"'] & \mathbb{R} \\ P \end{tikzcd} \tag{127}$$

where

$$\begin{aligned} \eta\colon & P \times \mathbb{R}_+ \times \mathbb{R}_+ \to P \\ & : (x_1, x_2, p_1, p_2, \lambda_1, \lambda_2) \mapsto (x_1, x_2, p_1, p_2) \end{aligned} \tag{128}$$

is the canonical projection and

$$
\begin{aligned}
G &: P \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \\
&: (x_1, x_2, p_1, p_2, \lambda_1, \lambda_2) \mapsto \lambda_1(\|p_1\| - \overline{m}_1) + \lambda_2(\|p_2\| - \overline{m}_2).
\end{aligned} \tag{129}
$$

The system D is the set of vectors $(x_1, x_2, p_1, p_2, \dot{x}_1, \dot{x}_2, \dot{p}_1, \dot{p}_2)$ in $M \times M \times V^* \times V^* \times V \times V \times V^* \times V^*$ satisfying relations

$$\langle x_2 - x_1, g(x_2 - x_1)\rangle < 0 \tag{130}$$

$$\langle \dot{x}_1, g(\dot{x}_1)\rangle > 0 \qquad \langle \dot{x}_2, g(\dot{x}_2)\rangle > 0 \tag{131}$$

$$p_1 = \overline{m}_1 \frac{g(\dot{x}_1)}{\|\dot{x}_1\|} \tag{132}$$

$$p_2 = \overline{m}_2 \frac{g(\dot{x}_2)}{\|\dot{x}_2\|} \tag{133}$$

$$\dot{p}_1 = \frac{\mathrm{D}U(\|x_2 - x_1\|)}{2\|x_2 - x_1\|}\left(\frac{\|\dot{x}_1\|}{\overline{m}_1} + \frac{\|\dot{x}_2\|}{\overline{m}_2}\right) g(x_2 - x_1) \tag{134}$$

$$\dot{p}_1 + \dot{p}_2 = 0. \tag{135}$$

The system D is not integrable. The initial constraint set C^0 is the set of covectors $(x_1, x_2, p_1, p_2) \in P$ satisfying equations

$$\|p_1\| = \overline{m}_1 \tag{136}$$

$$\|p_2\| = \overline{m}_2. \tag{137}$$

The tangent set $\mathsf{T}C^0$ is the set of vectors $(x_1, x_2, p_1, p_2, \dot{x}_1, \dot{x}_2, \dot{p}_1, \dot{p}_2) \in P \times V \times V \times V^* \times V^*$ satisfying equations

$$\|p_1\| = \overline{m}_1 \tag{138}$$

$$\|p_2\| = \overline{m}_2 \tag{139}$$

$$2\langle g^{-1}(p_1), \dot{p}_1\rangle + \frac{\mathrm{D}U(\|x_2 - x_1\|)}{\|x_2 - x_1\|}\langle \dot{x}_2 - \dot{x}_1, g(x_2 - x_1)\rangle = 0 \tag{140}$$

$$2\langle g^{-1}(p_2), \dot{p}_2\rangle + \frac{\mathrm{D}U(\|x_2 - x_1\|)}{\|x_2 - x_1\|}\langle \dot{x}_2 - \dot{x}_1, g(x_2 - x_1)\rangle = 0. \tag{141}$$

The set $\mathsf{T}^\P C^0$ is composed of vectors $(x_1, x_2, p_1, p_2, \delta x_1, \delta x_2, \delta p_1, \delta p_2) \in P \times V \times V \times V^* \times V^*$ satisfying equations

$$\|p_1\| = \overline{m}_1 \tag{142}$$

$$\|p_2\| = \overline{m}_2 \tag{143}$$

$$\delta x_1 = 2k_1 g^{-1}(p_1) \tag{144}$$

$$\delta x_2 = 2k_2 g^{-1}(p_2) \tag{145}$$

$$\delta p_1 = (k_1 + k_2)\frac{\mathrm{D}U(\|x_2 - x_1\|)}{2\|x_2 - x_1\|}\, g(x_2 - x_1) \tag{146}$$

and

$$\delta p_1 + \delta p_2 = 0 \tag{147}$$

with arbitrary values of the parameters k_1 and k_2.

The constraint set C^1 is the set of covectors $(x_1, x_2, p_1, p_2) \in P$ satisfying the equation

$$\langle x_2 - x_1, p_1 + p_2\rangle = 0 \tag{148}$$

in addition to relations (136) and (137).

The tangent set $\mathsf{T}C^1$ is the set of vectors $(x_1, x_2, p_1, p_2, \dot{x}_1, \dot{x}_2, \dot{p}_1, \dot{p}_2) \in P \times V \times V \times V^* \times V^*$ satisfying equations

$$\langle x_2 - x_1, p_1 + p_2 \rangle = 0 \tag{149}$$

and

$$\langle \dot{x}_2 - \dot{x}_1, p_1 + p_2 \rangle + \langle x_2 - x_1, \dot{p}_1 + \dot{p}_2 \rangle = 0 \tag{150}$$

in addition to equations (138)–(141).

The set $\mathsf{T}^{\P}C^1$ is composed of vectors $(x_1, x_2, p_1, p_2, \delta x_1, \delta x_2, \delta p_1, \delta p_2) \in P \times V \times V \times V^* \times V^*$ satisfying equations

$$\delta x_1 = 2k_1 g^{-1}(p_1) + k_3(x_2 - x_1) \tag{151}$$

$$\delta x_2 = 2k_2 g^{-1}(p_2) + k_3(x_2 - x_1) \tag{152}$$

$$\delta p_1 = (k_1 + k_2)\frac{\mathrm{D}U(\|x_2 - x_1\|)}{2\|x_2 - x_1\|}\, g(x_2 - x_1) + k_3(p_1 + p_2) \tag{153}$$

and

$$\delta p_1 + \delta p_2 = 0 \tag{154}$$

with arbitrary values of the parameters k_1, k_2 and k_3 in addition to equations (142) and (143).

The set C^2 is the same as C^1. Hence $\overline{C} = C^1$. The integrable part $\overline{D}$ of D is the set of vectors $(x_1, x_2, p_1, p_2, \dot{x}_1, \dot{x}_2, \dot{p}_1, \dot{p}_2)$ in $M \times M \times V^* \times V^* \times V \times V \times V^* \times V^*$ satisfying relations

$$\langle x_2 - x_1, g(x_2 - x_1) \rangle < 0 \tag{155}$$

$$\langle \dot{x}_1, g(\dot{x}_1) \rangle > 0 \qquad \langle \dot{x}_2, g(\dot{x}_2) \rangle > 0 \tag{156}$$

$$p_1 = \overline{m}_1 \frac{g(\dot{x}_1)}{\|\dot{x}_1\|} \tag{157}$$

$$p_2 = \overline{m}_2 \frac{g(\dot{x}_2)}{\|\dot{x}_2\|} \tag{158}$$

$$\dot{p}_1 = \frac{\mathrm{D}U(\|x_2 - x_1\|)}{2\|x_2 - x_1\|}\left(\frac{\|\dot{x}_1\|}{\overline{m}_1} + \frac{\|\dot{x}_2\|}{\overline{m}_2}\right) g(x_2 - x_1) \tag{159}$$

$$\dot{p}_1 + \dot{p}_2 = 0 \tag{160}$$

and

$$\langle \dot{x}_2 - \dot{x}_1, p_1 + p_2 \rangle = 0. \tag{161}$$

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